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Equal versus Differential Weighting for Multiattribute Decisions: There Are No Free Lunches

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Recent work on parameter insensitivity in linear models ("equal weighting" arguments) is examined for its implications for multiattribute decision making. [The key factor for the case equal weighting.] It is argued that the non-negative attribute intercorrelations upon which the case for equal weighting of attributes is based does not generally hold for multi-attribute decisions because the very tradeoffs which create the decision "problem" imply negative intercorrelations. After an examination of the → next page

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cont.

likely values of such negative intercorrelations, the effects of using "incorrect" weights in multiattribute decision making on the correlation between true and estimated evaluations and on the expected utility loss due to use of incorrect weights are evaluated. Implications of the theoretical results are discussed in relation to the precision with which multiattribute methods must assess the weights. Suggestions are included for how to use the theorems in this paper for determining the required accuracy of weight estimation for any gived applied problem.

Equal versus Differential Weighting for Multiattribute Decisions: There are No Free Lunches

The task of evaluating several alternatives each varying on a number of attributes or dimensions is a problem confronting anyone who makes decisions. Such tasks are often referred to as multiattribute, multidimensional or multiple-cue decision problems. Examples include diagnoses by clinical psychologists, fund allocations by public officials, and transportation mode choices by commuters. Economists, management scientists, decision theorists, and psychologists have developed many different techniques to aid decision makers faced with multiattribute decision problems.

Although a wide diversity of techniques have been recommended for resolving multiattribute decision problems, most are ultimately based on a weighted linear model. The essential steps of this general technique are (a) identifying the relevant attributes for evaluating the alternatives, (b) locating each alternative on each attribute dimension, (c) scaling the value or utility of those attribute locations, obtaining importance weights for each attribute, and aggregating the importance weights and attribute values into an overall evaluation of each alternative. The decision is then easy: the alternative with the highest aggregate value is selected.

The primary procedural difference among the various multiattribute decision aids based on weighted linear models is in how the attribute importance weights are determined. Thus, the question of which multiattribute decision aid to employ becomes a question of which procedure produces the "best" attribute weights.

A number of theoretical and empirical studies (Dawes & Corrigan, 1974; Einhorn & Hogarth, 1975; Schmidt, 1971; Wainer, 1976; and Wilks, 1938) have demonstrated that the problem of determining optimal weights is in many instances moot because all possible sets of weights must produce essentially the same results. Because, in Wainer's (1976) words, "it don't make no nevermind" what weights are used, many authors suggest using the simplest possible procedure: don't estimate weights, just give equal weight to each attribute.

While the conditions of the "equal weighting" theorems are quite general, the argument that equal (or even arbitrary) weights can be substituted without loss for optimal weights was developed within the context of multiple regression analysis rather than multiattribute decision making. Newman (Note 1) has shown that for at least some multiattribute decision problems the specific weights used have a substantial effect on the resulting evaluations. The present paper includes (a) a demonstration that the major premise of the equal weights argument is inappropriate for virtually all multiattribute decision problems, (b) an investigation of the correlation between evaluations based on optimal weights and those based on any other specified weights (including equal weights), (c) an examination of the loss in utility or value resulting from use of nonoptimal weights, and (d) a discussion of the importance of accurate estimation of optimal weights under various conditions.

Before the issue of the appropriateness of the equal weights argument for multiattribute decisions is addressed, the argument itself is reviewed briefly.

The Case for Equal Weighting

In this section are presented the theoretical results underlying the equal weights argument, its history, and its major limiting condition. The notation to be used throughout the paper is presented first.

The multiattribute utility of an alternative *i* is given by

$$y_i = \sum_{j=1}^{n} w_j x_{ij} \tag{1}$$

where x_{ij} is the utility of alternative *i* on attribute *j*, w_j is the weight accorded attribute *j* in the overall evaluation, and y_i is the aggregated utility. When Equation 1 is actually used to compare alternatives in a multiattribute decision problem, estimates of the attribute weights w_j will not in general exactly equal the true weights. Thus in practice estimated weights $\hat{w}_j$ are used to give $\hat{y}_i$, the estimated aggregate utility:[1]

$$\hat{y}_i = \Sigma \hat{w}_j x_{ij}. \tag{2}$$

In two-attribute problems (more general problems are considered in a later section) if we require the weights to sum to one (without loss of generality), the true weights can be represented by $\underline{w}$ and $1 - \underline{w}$ and the estimated weights by $\hat{\underline{w}}$ and $1 - \hat{\underline{w}}$. The question of interest, then, is: what is the correlation between $\underline{y}$ (using $\underline{w}$) and $\hat{\underline{y}}$ (using $\hat{\underline{w}}$)? The answer depends on the inter-attribute correlation (i.e., the correlation between x_{i1} and x_{i2} across the alternatives) and is given by the following theorem. (Proofs of the mathematical results are outlined in the appendix.)

Theorem 1 If x_{i1} and x_{i2} are standardized so that their variances are equal, if $\rho_{x_1x_2}$ is their correlation, if y and $\hat{y}$ are as defined in Equations 1 and 2, and if $\rho_{y\hat{y}}$ represents their correlation, then

$$\rho_{y\hat{y}} = \frac{1 + (2w\hat{w}-w-\hat{w})(1-\rho_{x_1x_2})}{\{[1-2(1-\rho_{x_1x_2})w(1-w)][1-2(1-\rho_{x_1x_2})\hat{w}(1-\hat{w})]\}^{1/2}} . \tag{3}$$

The equal weights argument--that equal weights can be substituted for the true weights without much loss in correlation--is easily derived from Equation 3 by using $\hat{w} = .5$. Let $\hat{y}_e$ be the estimate of the aggregate utility using equal weights. The correlation between y and $\hat{y}_e$ is given by the following corollary to Theorem 1.

Corollary 1.1. Given the conditions of Theorem 1 and if $\hat{w} = .5$, then

$$\rho_{y\hat{y}_e} = \left[\frac{.5(1+\rho_{x_1x_2})}{1-2(1-\rho_{x_1x_2})w(1-w)}\right]^{1/2} \tag{4}$$

Insert Figure 1 about here

Figure 1 shows $\rho_{y\hat{y}_e}$ as a function of the true weight w for selected values of $\rho_{x_1x_2}$. It is clear from Figure 1 that $\rho_{y\hat{y}_e}$ is always relatively high (greater than $\sqrt{.5} = .71$) as long as $\rho_{x_1x_2}$ is not negative. Even when $\rho_{x_1x_2}$ is zero, the true weights must be very extreme (near 0 and 1) before $\rho_{y\hat{y}_e}$ is substantially reduced. For example, when $\rho_{x_1x_2}$ equals zero and one of the true weights is three times larger than the other, using equal

weights still produces a correlation between y and $\hat{y}_e$ of 0.89; for a five to one ratio of true weights, the equal weights correlation is 0.83. Further, the equal weights correlation is substantial ($\geq .60$) even when there is a large negative ($\leq -.75$) correlation between the two attributes.

To summarize the equal weighting argument, if the correlation between the attributes is nonnegative then the correlation between the true evaluations of the alternatives and the estimated evaluations using equal weights is at least .71. Further, this correlation is substantially higher whenever the true weights are in the neighborhood of equal weights and/or whenever the correlation between the attributes is large (and positive).

The equal weighting argument has a long history. Wilks (1938) proved in the context of mental testing that given a set of reasonably general conditions and a large number of intercorrelated variables, any linear combination (i.e., arbitrary weights) has essentially the same value as any other. Gulliksen (1950), also within a testing framework, provided a more general formulation yielding the same conclusion. Einhorn and Hogarth (1975) used a result due to Ghiselli (1964) to derive a lower bound for $\rho_{y\hat{y}_e}$. For two attributes, the Einhorn and Hogarth lower bound is easily derived from Equation 3 (Corollary 1.1) by using $\underline{w}$ = 1 or 0. Because this lower bound is surprisingly high for positive intercorrelations, Einhorn and Hogarth suggest that for most regression problems--whenever scoring of independent variables can be reversed so that all correlations with the dependent variable are positive--it is not worth the computational time or the loss of degrees of freedom in a statistical analysis to estimate regression weights. Green (1977) and Wainer (1976) give a result equivalent to Theorem 1, the only difference being that they constrain the weights by

standardizing $\underline{y}$ and $\hat{\underline{y}}$ to have unit variance rather than by requiring the weights to sum to one. Wainer further shows that if the actual weights are uniformly distributed over the interval [.25, .75] then the expected loss in proportion of variance accounted for due to using equal weights in a two-attribute problem is at most 2.1%.

The implication for multiattribute decision procedures from these studies is clear. If the attributes are positively intercorrelated a multiattribute model with equal weights will have a very high correlation with a model based on the decision maker's actual weights no matter what they might be. Thus if it is possible to formulate the decision problem so that the attributes are positively intercorrelated, there is no need to estimate weights. Only if the correlations are negative (and even then in the two attribute case the correlation must be rather close to -1) is it necessary to use estimated weights instead of equal weights. Therefore, the important question becomes: Is it possible to formulate a typical multiattribute decision problem so that there are no (large) negative intercorrelations, or at least so that the average intercorrelation is positive? This question is the topic of the next section.

Are Attribute Intercorrelations Positive in Multivariate Decision Problems?

In some decision problems the attributes do seem to be positively correlated. For example, Dawes (1971) and many others have applied linear models to the problem of selecting graduate students on the basis of scores on the Graduate Record Exam (GRE) and undergraduate grade point average (GPA). Because both GRE scores and GPA measure academic achievement and ability, they tend to be highly correlated. As a consequence of Theorem 1, any two

linear functions utilizing GRE scores and GPA will produce highly correlated results (see Dawes & Corrigan, 1974, and Goldberg, 1977, for demonstrations of this fact using actual admissions data).

However, such positive correlations among attributes are not characteristic of typical multiattribute decision problems, and even in the case of the graduate student selection problem discussed above the positive correlation is only apparent and not reflective of the true decision problem. The remainder of this section argues that attribute intercorrelations in multiattribute decision problems are inherently negative and hence the case for equal weights considered in the previous section does not apply.

The essence of a multiattribute decision problem is that the several attribute dimensions considered are not simultaneously maximized in a single alternative. In other words, the decision maker must make tradeoffs, trading an improved standing on one attribute for a diminished standing on others. Indeed, multiattribute utility analysis is often referred to as "tradeoff analysis." If it were not for the necessity of tradeoffs, there would be no "problem" because anyone (and any linear model regardless of its weights) would surely pick the alternative with maximum values on all the attributes. For example, any graduate admissions committee knows whether to admit a student with a 700 GRE score and a 3.8 GPA or a student with a 400 GRE score and a 2.5 GPA; it is the students with high GRE scores and low GPA or vice versa who create a decision problem.

Tradeoffs among attributes inherently induce negative attribute intercorrelations. Newman, Seaver, and Edwards (Note 2) cite two reasons why tradeoffs, and hence negative attribute intercorrelations, are to be expected in virtually every multiattribute decision problem. First, it is a simple fact of life that one good thing must often be exchanged for

another good thing. ("There's no such thing as a free lunch.") For example, a person selecting a transportation mode from home to work who desires to minimize both cost and travel time finds that these attributes are negatively correlated within the set of available transportation alternatives (e.g., walking, bicycle, mass transit, carpool, and private car). In an applied problem of selecting handgun ammunition for the Denver police department, Hammond and Adelman (1976) found that desirable bullet attributes--stopping effectiveness and inability to produce severe injury or death--were substantially (but not perfectly) negatively correlated. Similar examples are typical in published accounts of multiattribute decision problems.

Second, even if the environmental attribute intercorrelations are not negative (as in the GRE-GPA example), these correlations will be negative among the admissible (i.e., nondominated) alternatives. This is illustrated for two attributes in Figure 2. Assuming that increasing amounts of each attribute are desirable, only the alternatives connected by the line segments are admissible; all other alternatives are dominated. That is, for any alternative not on the boundary (often called the Pareto frontier due to Pareto's [1907] use of the dominance principle in social welfare economics) there exists an alternative on the boundary at least as good on both attributes. Any rational decision maker would prefer an alternative on the Pareto frontier to any alternative in the interior of Figure 2. While the case for three or more attributes is not as easily depicted graphically, it is exactly analogous logically.

Insert Figure 2 about here

Therefore, the answer to the question posed in this section--are attribute intercorrelations positive in multiattribute decision problems--is no. However, this does not mean that the equal weights argument is always inapplicable for such problems, for the loss due to use of nonoptimal weights is not substantial except when the interattribute correlations are both negative and large.

In the preceding sections it has been established that (a) the correlation between evaluations using optimal weights ($\underline{y}$) and evaluations based on equal weights ($\hat{y}_e$) is a function of the interattribute correlation $\rho_{x_1x_2}$ (Corollary 1.1) and (b) only those alternatives on the Pareto frontier are within the admissible decision set. Given these two results, the question of the applicability of the equal weights argument for multiattribute decision problems reduces to finding $\rho_{x_1x_2}$ for those alternatives on the boundary.

In two-attribute problems, x_1 and x_2 must have a rank-order correlation of -1.0 among the nondominated alternatives. Even given this constraint, sets of nondominated alternatives can be constructed in which $\rho_{x_1x_2}$ ranges from 0.0 to -1.0. In the next section it is demonstrated that $\rho_{x_1x_2}$ actually ranges only from -.7 to -1.0 for typical or likely multiattribute problems, and a set of prototypical Pareto frontiers is constructed for use in the remainder of the paper.

Interattribute Correlations in Typical Multiattribute Decision Problems

The equation $x_1^a + x_2^a = 1$ describes a set of curves which are symmetric, relatively tractable, and fit the Pareto conditions. This set will be used throughout the remainder of the paper to represent prototypical Pareto frontiers or boundaries. Several curves from the

set are illustrated in Figure 3. For convenience, the highest value of x_i is set equal to one, the lowest to zero.

Insert Figure 3 about here

When the exponent a equals 1, the Pareto frontier is a line representing a severe tradeoff relationship: a gain in standing on one attribute is accompanied by loss of an equivalent amount on the other attribute. For large values of a (e.g., a = 8), the tradeoff relationship is very weak: a low or moderate value for either attribute can be substantially improved with only a small loss in the value of the other attribute. Curves with large exponents are hardly typical of practical multiattribute decision problems because in such cases an alternative exists with near maximum values on both attributes (e.g., for a = 8, an alternative exists with a value of .92 on both attributes). If an alternative with almost maximum values on both alternatives exists, there is no decision "problem" and none of the various multiattribute decision aids would be used anyway. Thus, the Pareto prototypical curve for a = 8 serves as an upper bound for the Pareto curves likely to be encountered in practice.

The prototypical curve with a = 1 serves as a lower bound for similar reasons. The concave curves produced by exponents less than one represent extreme tradeoff relationships, in which a small improvement in standing on either attribute requires a major reduction in the other. If a linear multiattribute function (i.e., Equation 1) is used to evaluate the alternatives along such a curve, then, regardless of weights, the most desirable alternative must have the maximum value on one attribute and the minimum value on the other. That is, when a is less than one, the decision problem reduces to a

choice between the two alternatives with a value of one on one alternative and a value of zero on the other. Because the machinery of multiattribute decision making does not seem particularly appropriate for aiding such decisions, the curve for $\underline{a} = 1$ serves as a lower bound for the types of Pareto frontiers to which one is likely to apply the techniques of multiattribute decision aids.

The above argument indicates that prototype curves with exponents between 1 and 8 can be used to represent most commonly encountered Pareto frontiers. Given this, $\rho_{x_1x_2}$ can be calculated as a function of the exponent $\underline{a}$. Obviously for $\underline{a} = 1$, $\rho_{x_1x_2} = -1$. Values of $\rho_{x_1x_2}$ for other exponents were estimated using Monte Carlo simulations with 5000 random alternatives distributed uniformly along the Pareto boundary. For moderate tradeoff relationships, $\underline{a} = 2$ and 3, $\rho_{x_1x_2} = -.93$ and $-.86$, respectively.[2] Even for a very weak tradeoff relationship, $\underline{a} = 8$, the interattribute correlation is -.73 which (if these prototypical curves are accepted as being representative of Pareto boundaries encountered in practice) establishes a reasonable upper bound for $\rho_{x_1x_2}$ in two-attribute decision problems.

Two practical examples reinforce the notion that in typical two-attribute decision problems the interattribute correlation ranges from -.7 to -1.0. In the Hammond and Adelman (1976) evaluation of handgun ammunition the correlation between stopping effectiveness and probability of injury and death was -.98 (when both were coded to correlate positively with desirability). Similarly, data assembled by Newman (Note 1) show automobile quality and cost to be correlated -.87 among the eight nondominated alternatives. The reader is invited to attempt construction of a reasonable alternative set with an interattribute correlation smaller than -.7. Of course, in any

actual example $\rho_{x_1x_2}$ can (and should) be computed precisely, making the issue of "typical" interattribute correlations moot.

Effects of Large Negative Interattribute Correlations on $\rho_{y\hat{y}}$

Equation 3 can be used to calculate the correlation between evaluations based on any specified true weights and those based on any specified estimated weights for a given $\rho_{x_1x_2}$. Of special interest are the results for estimations with $\hat{w}$ = .5 (equal weights) and $\hat{w}$ = 1 (extreme weights).

Figure 1 shows the effect on $\rho_{y\hat{y}_e}$ of the interattribute correlation $\rho_{x_1x_2}$ as a function of the optimal weight w. The curve shown for $\rho_{x_1x_2}$ = -.75--the reasonable upper bound on $\rho_{x_1x_2}$--approximates the maximum attainable correlation between evaluations based on equal and optimal weights in two-attribute decision problems. Clearly, when only nondominated alternatives are considered even a slight deviation of the true weights from equality causes a sharp reduction in the correlation between y and $\hat{y}_e$. For example, for a = 2 ($\rho_{x_1x_2}$ = -.93), the equal weights model would account for only 12.4% of the variance in the evaluative ratings of a decision maker whose true weights were .75 and .25. Using an equal weighting scheme in such situations would be a serious mistake.

The unsuitability of equal weights can also be demonstrated by examining how far the true weight w must deviate from .5 before it is better to use $\hat{w}$ = 1 instead of $\hat{w}$ = .5. That "breakpoint" weight is given by the following additional corollary to Theorem 1.

Corollary 1.2. Given the conditions of Theorem 1, if $\underline{w}$ is the larger of the two weights, then $\rho_{y\hat{y}}$ for $\hat{\underline{w}} = 1$ is greater than $\rho_{y\hat{y}_e}$ whenever

$$w > \frac{-\rho_{x_1x_2} + \sqrt{(1 + \rho_{x_1x_2})/2}}{(1 - \rho_{x_1x_2})} \tag{5}$$

This means, for example, that if the Pareto curve is approximately like the one for $\underline{a} = 2$ ($\rho_{x_1x_2} = -.93$), then it is better (in terms of $\rho_{y\hat{y}}$) to estimate $\hat{\underline{w}}$ as 1 than to use equal weights whenever the actual weight $\underline{w}$ is greater than .58! In other words, Corollary 1.2 suggests that if the rank order of the two weights can be determined, then for many two-attribute decision problems it is better to use extreme weights (1 and 0) in the weighted linear model than to use equal weights. This conclusion for decision problems is the antithesis of the equal weights argument discussed above.

Loss in Utility Due to Use of Nonoptimal Weights

The suitability of equal weighting in multiattribute decision problems was evaluated in the preceding sections by reference to the correlation $\rho_{y\hat{y}}$ between sets of evaluations based on actual weights and those based on various estimated weights. This method was used in order to parallel the arguments for equal weights made in earlier papers and to develop the basic premises of this paper in a familiar context. However, because the goal of a multiattribute decision procedure is a decision, not a set of evaluations, the correlation $\rho_{y\hat{y}}$ is not strictly adequate to assess the suitability of equal weights in this context. In this section a more

appropriate measure is developed and used to evaluate the equal weighting model.

The goal in making a multiple attribute decision is not to predict $\underline{y}$ with $\hat{\underline{y}}$ for all nondominated alternatives. Rather, the goal is to select the alternative (or at most a few alternatives) for which $\underline{y}$ is maximum. Unless $\underline{w}$ equals $\hat{\underline{w}}$, $\hat{\underline{y}}$ will not generally attain its maximum at the same alternative as will $\underline{y}$, the true function. Consequently, the decision maker will not obtain as great a total utility as would be possible if the true weights were used to select the alternative. These considerations lead to an appropriate, simple measure. If $\hat{\underline{i}}$ is the alternative for which $\hat{\underline{y}}$ is maximum, then $u_{y\hat{y}} = (y_{\hat{i}}/\max y)$ is the proportion of the maximum possible utility returned by using $\hat{\underline{y}}$. Thus, $u_{y\hat{y}}$ measures the adequacy of a given $\hat{\underline{y}}$ as a solution to a multiattribute decision problem for which the true weights are known. If $u_{y\hat{y}}$ is near one, then use $\hat{\underline{y}}$ is satisfactory because essentially the same aggregate utility is obtained regardless of whether $\underline{y}$ and $\hat{\underline{y}}$ select the same alternative. If $u_{y\hat{y}}$ is much below one, $\hat{\underline{y}}$ is not a satisfactory substitute for $\underline{y}$, and an attempt to estimate weights more accurately would be warranted.

The importance of a given $u_{y\hat{y}}$ is, of course, a function of the situation at hand: to the question "How low must $u_{y\hat{y}}$ be before rejecting $\hat{\underline{y}}$?" there can be no general answer. While values of $u_{y\hat{y}}$ above .95 would probably be considered satisfactory in most cases, even a 5% loss in utility might be considered unacceptable in decisions involving large amounts of time, great expense, or the health and safety of many people. Expenditures of additional time and money to estimate the weights more accurately would then be justified. On the other hand, if the

measures of the attribute values for each alternative were very unreliable, there might be so much error in the model that even a 20% utility loss would be inconsequential. Whatever, the purpose of this section is to illustrate what happens to $u_{y\hat{y}}$ when equal or other nonoptimal weights are used. Individual users of multiattribute decision aids can decide for themselves how much loss of utility can be tolerated in specific applications and then use the following information about $u_{y\hat{y}}$ to determine whether equal weighting (or any other scheme) is suitable for their decision problems.

The values $y_{\hat{i}}$ and max $\underline{y}$ cannot be determined except in the context of a specific set of nondominated alternatives or an explicitly described Pareto boundary. Therefore, the set of prototypical Pareto frontiers described by the function $x_1{}^a + x_2{}^a = 1$ is used to illustrate the effects on $u_{y\hat{y}}$ of various weight estimates. As above, the general result is presented first, followed by the equal- and extreme- weights cases. An illustrative example precedes the theoretical arguments.

A decision maker must choose an alternative from a large set of alternatives described by two attributes. The attributes are constrained so that all nondominated alternatives fall on a circular arc (i.e., $x_1^2 + x_2^2 = 1$; see the curve for $\underline{a} = 2$ in Figure 3). If the optimal or true weights are .75 and .25, respectively, for attributes 1 and 2, then the alternative with the maximum utility is the alternative (.95, .32), with (.75 • .95) + (.25 • .32) = .79. Use of equal weights would result in selection of the alternative (.71, .71). In terms of the assumed optimal or true weights, this alternative has utility (.75 • .71) + (.25 • .71) = .71. Thus, use of equal weights in the selection procedure yields only .71/.79 = 90% of the maximum possible utility. If the procedure used to estimate the weights

erred so badly as to reverse the weights (i.e., estimate the first weight as .25 and the second as .75), then the alternative (.32, .95) would be chosen. Its utility (in terms of the true weights) is only .47--a loss in utility of 40%. Thus when the task is to choose one alternative from the Pareto set, use of nonoptimal weights can cause major losses in utility; the utility obtained in a multiattribute decision problem is sensitive to inaccurate estimates of the parameters (weights). The above illustration is formalized in the following theorem.

Theorem 2. If the attributes are constrained so that $x_1^a + x_2^a = 1$, then

$$u_{y\hat{y}} = \frac{w[(\frac{1-\hat{w}}{\hat{w}})^{\frac{a}{a-1}} + 1]^{-1/a} + (1-w)[(\frac{\hat{w}}{1-\hat{w}})^{\frac{a}{a-1}} + 1]^{-1/a}}{w[(\frac{1-w}{w})^{\frac{a}{a-1}} + 1]^{-1/a} + (1-w)[(\frac{w}{1-w})^{\frac{a}{a-1}} + 1]^{-1/a}} \qquad (6)$$

Corollary 2.1. Given the same conditions as in Theorem 2, $u_{y\hat{y}_e}$, the proportion of the maximum possible utility obtained by using equal weights, is given by

$$u_{y\hat{y}_e} = \frac{2^{-1/a}}{w[(\frac{1-w}{w})^{\frac{a}{a+1}} + 1]^{-1/a} + (1-w)[(\frac{w}{1-w})^{\frac{a}{a+1}} + 1]^{-1/a}} . \qquad (7)$$

Figure 4 shows $u_{y\hat{y}_e}$ as a function of w for selected values of a. For weak tradeoff relationships (a near 8), using equal weights to select the

alternative returns at least 91.7% of the maximum possible utility; when the true weights are between .1 and .9, at least 95% of the maximum possible utility is obtained. For severe tradeoffs (a near 1), using equal weights to estimate true weights that are even slightly different from equality results in substantial losses in utility. For example, if the true weights are .4 and .6 and a = 1.1, an alternative with only 88.7% of the maximum possible utility is selected when equal weights are used. For other true weights as much as 46.7% of the utility may be lost by using equal weights. Use of equal weights for selection in intermediate tradeoff relationships (a = 2 or 3) can also result in clear loss of utility.

Insert Figure 4 about here

The impact of these results is further demonstrated by determining the "breakpoint" weight for which extreme weights (1 and 0) produce the same value of $u_{y\hat{y}}$ as do equal weights. This breakpoint weight is given as a function of a by the following corollary.

Corollary 2.2. Given the conditions of Theorem 2, if w is the larger of the two weights, then $u_{y\hat{y}}$ for $\hat{w} = 1$ is greater than $u_{y\hat{y}_e}$ whenever

$$w > 2^{-1/a} . \tag{8}$$

Thus, when a = 1.1, it is better to use extreme weights than equal weights unless the true weights are between .47 and .53. That is, for sharp tradeoff relationships, it is almost always better to estimate the larger weight

as 1 and the smaller as 0 rather than to estimate both as .5. The breakpoint weight for $\underline{a}$ = 2 is .71. For weak tradeoff relationships ($\underline{a}$ = 8) the breakpoint is .92, indicating that in such cases equal weights are generally better than extreme weights.

The results on loss of utility caused by using equal weights and the superiority of extreme weights over equal weights for many tradeoff relationships mean that routine use of equal weighting in multiattribute decision problems is inadvisable. The question then becomes one of specifying how close to the true weights the estimates need to be in order to avoid substantial utility losses. The answer lies in Equation 6. The utility obtained when using $\hat{\underline{w}}$ rather than $\underline{w}$ to select the alternative is plotted as a percent of maximum possible utility in Figures 5, 6, and 7 for $\underline{a}$ = 1.1, 2, and 8, respectively to illustrate the effect of inaccurate estimates for different tradeoff relationships.

Insert Figures 5, 6, and 7 about here

The marked differences among Figures 5, 6, and 7 illustrate that choice of a particular strategy for estimating weights in two-attribute utility problems should be guided by the nature of the interattribute relation. For very weak trade-offs ($\underline{a} \sim 8$), use of equal weights results in utility losses of less than 10% no matter what the true weights; here use of equal weights is appropriate unless the true weights are thought to be quite extreme and a high degree of accuracy is desired. For severe tradeoff relationships ($\underline{a} \sim 1.1$), that the weights be in the correct rank order is absolutely critical; given that, extreme weights (i.e., 0, 1) result in losses of utility of less than 10% no matter what the true weights are. For moderate tradeoff relationships ($\underline{a} \sim 2$), use of neither equal

nor extreme weights is generally appropriate. Use of equal weights causes a loss of utility of 10% or more if $\underline{w} \leq .25$ (or $\underline{w} \geq .75$), whereas use of extreme weights does so if $.33 \leq \underline{w} \leq .67$. It is with moderate tradeoff relations, then, that accurate weight estimation is most important, and moderate tradeoffs are probably the most likely to be encountered in practice. However, as long as the rank order is correct, the estimates still do not need to be very accurate. Any estimate between .1 and .4, for example, produces at least 95% of the maximum possible utility for a true weight of .25 when $\underline{a} = 2$.

Generalization to Three or More Attributes

Although the discussion to this point has been restricted to decision problems with two attributes, all formal results and theorems except for the "breakpoint" corollaries can be easily generalized to problems with three or more attributes. For example, many mathematical statistics texts present generalizations of Theorem 1 (e.g., Rao, 1965, p. 441; and Ghiselli 1964, pp. 306-309). However, such formal generalizations are not particularly useful because systematic analyses (as in Figures 1 to 7) become impossibly unwieldy when three or more attributes are considered. This section includes (a) a demonstration that equal weighting is inappropriate for general multiattribute decision problems because average interattribute correlations are necessarily negative when the alternatives form a Pareto surface, and (b) derivation of an upper bound for $u_{y\hat{y}_e}$.

Average interattribute correlations. When only nondominated alternatives--those on the Pareto frontier--are considered, a gain in standing on one attribute must be compensated by a reduced standing on

other attributes, thereby inducing negative interattribute correlations. However, because the necessary compensation for a gain for one attribute can be spread over many attributes, the interattribute correlations are not of such great magnitude as those in two-attribute problems. For example, if all interattribute correlations are equal, their value must lie in the interval (0, -1/[k-1]) for k attributes. The nonzero boundary of this interval ranges from -1.00 for two attributes to -.14 for eight.

Likely values for the average interattribute correlation may be estimated by examining protypical Pareto surfaces of the same type considered for two attributes. These surfaces are defined by $\Sigma x_i^a = 1$; again _a_ near 1 represent severe tradeoffs and large _a_ represent weak tradeoffs. Figure 8 depicts the average interattribute correlation (determined by Monte Carlo simulation) as a function of the severity of the tradeoff relationship (_a_) and the number of attributes (_k_). Except for two attributes the degree of the tradeoff relationship has surprisingly little effect on $\bar{\rho}_{x_i x_j}$: no matter what the value of _a_, $\rho_{x_i x_j}$ is very near its boundary value for each _k_. That is, when the alternatives are on a Pareto surface the average interattribute correlation is likely to be as negative as possible given the number of attributes _k_.

Insert Figure 8 about here

Einhorn and Hogarth (1975, p. 175) show that the minimum possible squared correlation between y and $\hat{y}_e$ is given by

$$\rho^2_{y\hat{y}_e} = \frac{1 + (k-1)\rho}{k} \tag{9}$$

where ρ is the interattribute correlation for all attribute pairs. (Note that $\rho_{y\hat{y}_e}$ is a decreasing function of k.) Using the least negative value of ρ from Figure 8 ($\rho = -.11$ for $k = 8$ and $a = 8$) gives a minimum value of $\rho_{y\hat{y}_e}$ of only .03. Clearly, it is possible for an equally weighted composite of many attributes to be a very inaccurate estimator of the true aggregate value of nondominated alternatives.

Data from an actual application of a multiattribute decision aid demonstrate that the above problems are not merely hypothetical. Roche (Note 3, also see summary in Keeney & Raiffa, 1976, pp. 365-376) presents six alternative budget allocations considered by a Massachusetts school board which must distribute a fixed budget among four program areas (language arts, mathematics, science, and social studies). The set of alternative allocations is described by the protypical Pareto surface with $a = 1$. While the interattribute correlations are not all equal (they vary from -.96 to +.73), the average correlation is -.33, as predicted by Figure 8 ($a = 1$, $k = 4$).

The upper bound for $u_{y\hat{y}_e}$. Corollary 2.1 and Figure 4 define for two attributes the proportion of the maximum possible utility obtained by using equal weights. Such a formal comparison of all possible combinations of true weights to equal weights becomes impossible as the number of attributes increases. However, examination of the special case in which all true and estimated weights except one are equal is of interest; this case is described by Theorem 3.

Theorem 3. Assume that $w_2 = w_3 = \ldots w_k$ and that $\Sigma x_i^a = 1$. Then using estimates $\hat{w}_1$ and $\hat{w}_2 = \hat{w}_3 = \ldots \hat{w}_k = (1 - \hat{w}_1)/(k - 1)$ yields

$$u_{y\hat{y}} = \frac{w_1 f(\hat{w}_1) + (1 - w_1)g(\hat{w}_1)}{w_1 f(w_1) + (1 - w_1)g(w_1)} \tag{10}$$

where $f(w) = \{[(\frac{1 - w}{w}) (\frac{1}{k - 1})^{1/a}]^{\frac{a}{a - 1}} + 1\}^{-1/a}$

and $g(w) = [\frac{1 - f^a(w)}{k - 1}]^{1/a}$

Theorem 3 gives the ratio between the utility obtained by estimating w_1 and $\hat{w}_1$ and the maximum possible utility when all other weights are correctly estimated as equal to one another. Because the unreasonable assumption that weights w_2 through w_k are equal favors the equal weighting model, an upper bound on $u_{y\hat{y}_e}$ is given by setting $w_1 = \frac{1}{k}$ (and therefore $w_2 = w_3 = \ldots w_k = \frac{1}{k}$).

Corollary 3.1. If $\Sigma x_i^a = 1$ describes the Pareto surface and if $w_1 = w_2 = \ldots = w_k = \frac{1}{k}$ then

$$u_{y\hat{y}_e} \leq \frac{k^{-1/a}}{w_1 f(w_1) + (1 - w_1)g(w_1)} \tag{11}$$

Figure 9 shows the upper bound for a moderate tradeoff relationship (a = 2) as a function of the true weight w_1 for various values of k; Figure 10 illustrates the effect of various tradeoff relationships on the bound for $u_{y\hat{y}_e}$ for k = 4. As Figure 9 demonstrates, the potential for utility loss from using equal weights increases as k increases; these potential losses are large even when all true weights but one actually are equal. Thus routine use of equal weights in multiattribute decision problems is clearly not warranted.

Insert Figures 9 and 10 about here

Newman (Note 1) presents a multiattribute decision problem which demonstrates that the potential utility losses stemming from use of the equal weighting model can actually occur. Newman's example concerns an automobile evaluation system developed by the Automobile Club of Southern California. The Auto Club identified and specified weights for 11 automobile attributes. Newman applied the Auto Club weights and equal weights to the 11 attribute values for 24 automobiles. Use of equal weights results in selection of a car which (in terms of the Auto Club weighting system) has a value of only 59 compared to the value of 68 for the car selected by the Auto Club weights. Thus, use of equal weights results in a utility loss of 13%. Interestingly, use of rank weights (the most important attribute is assigned a weight of $\underline{k}$, the second most important $\underline{k}$ - 1, etc.) results in selection of the same automobile as does use of the Auto Club's actual weights, for a utility loss of zero.

Discussion

The key issue in this paper is not whether the equal weights argument of Einhorn and Hogarth (1975), Green (1977), Wainer (1976) and others is correct; the validity of their results is not questioned. What is questioned is the advisability of routine use of equal weights in multiattribute decision problems. Such use has been suggested by Dawes and Corrigan (1974) and Einhorn and McCoach (1977), but is questioned by Newman (Note 1).

The arguments developed in this paper demonstrate that the case for equal weights cannot be directly transferred from the multiple regression

context in which it was developed to the context of multiattribute decision problems. The primary basis for this conclusion is an important distinction between the goals of regression and multiattribute decision-making. Whereas the goal of regression is accurate prediction of the outcome or evaluation for all alternatives, the goal in a decision problem is selection of the one best alternative. This distinction has been blurred by the use of regression models ("bootstrapping," "policy capturing," etc.) as multiattribute decision aids (e.g., Dawes, 1971).

When the evaluation-selection distinction is made, consideration must be restricted to those alternatives on the Pareto boundary--the alternatives which have any chance of being selected. This restriction induces large negative correlations among the alternatives and leads to sharply lowered correlations between the evaluations produced by the true and estimated weights. Further, the goal of selecting the best alternative means that $\rho_{y\hat{y}}$ is not an appropriate measure of the adequacy of a set of estimated weights. A more suitable measure is $u_{y\hat{y}}$, the ratio of the utility obtained using the estimated weights to the maximum possible utility possible using the true weights.

The theoretical results based on the appropriate measure $u_{y\hat{y}}$ yield three important conclusions: (a) the routine use of equal weighting in multiattribute decision problems is not justifiable because equal weighting can cause important losses of utility, and (b) how much utility is lost due to use of equal weights depends not only on the true weights but also on the nature of the tradeoff relationship and on the number of attributes, and (c) if the rank order of the estimated weights is correct, inaccurate estimation leads to very small utility losses.

The conclusions about $u_{y\hat{y}}$ listed above, derived from Theorem 2 and Figures 4-7, assume an infinite set of alternatives on a protypical Pareto curve described by $\Sigma x_i^a = 1$. Although these assumptions are not unreasonable, they do not hold for all decision problems. However, for any particular problem $u_{y\hat{y}}$ can be computed directly for as many hypothesized sets of true and estimated weights as is desired. The results of such an analysis can then be used to evaluate the adequacy of the equal weights--or any other--model for that problem.

The procedure for calculating $u_{y\hat{y}}$ is straight forward: (a) determine the set of nondominated alternatives, (b) select sets of weights which cover the range of plausible true weightings, (c) find the best alternative and hence the maximum possible utility for each plausible weighting, (d) select sets of weights which cover a wide range of potential estimated weightings (these sets should include all those in [b]), (e) calculate $u_{y\hat{y}}$ for each pair of estimated and hypothesized true weightings (i.e., find the alternative which the set of estimated weights would select as best), (f) calculate its utility using the set of hypothesized true weights, and divide that utility by the maximum possible utility--determined in (c)--for that hypothesized weighting.

A practical example demonstrates the usefulness of calculating $u_{y\hat{y}}$ and also illustrates some of the differences between real decision problems and the idealized (infinite number of alternatives, prototypical Pareto boundaries) problems considered in the theoretical discussion. The example is from Hammond and Adelman (1976) and involves the selection of handgun ammunition for a metropolitan police department. The two bullet attributes most relevant for the decision were stopping effectiveness (the probability that a man shot in the torso would be incapable of

returning fire) and severity of injury (the probability that a man shot in the torso would die within two weeks of being shot). Hammond and Adelman used a panel of experts to locate 80 bullets on these two attributes (see their Figure 3). Only three of the 80 bullets are "efficient" alternatives (an alternative is efficient if it is on the Pareto boundary and is not dominated by a convex linear combination of other alternatives; see Coombs & Avrunin, 1977, for a further discussion of efficiency) with any chance of being selected by a linear model.

Because there are only three eligible alternatives, only three ranges of estimates for $\underline{w}$, the true weight on stopping effectiveness, need be considered, with each of the three ranges leading to selection of a different alternative. In this case, a true weight between 0 and .33 leads to selection of the bullet with minimum stopping effectiveness and minimum injury (represented by (0, 1) when the attributes are scaled from 0 to 1), between .33 and .54 to the alternative (.28, .86), and between .54 and 1 to the alternative (1, 0). With only three alternatives it is easy to calculate $u_{y\hat{y}}$. For example, (1, 0) would be the best alternative for a true weight on stopping effectiveness of .7, yielding a utility of .7. Incorrect use of equal weights would lead to selection of the (.28, .86 alternative, giving a utility of (.7 x .28) + (.3 x 86) = .45 and $u_{y\hat{y}} = .45/.7 = .65$.

All possible values for $u_{y\hat{y}}$ are plotted in Figure 11. This figure shows that (a) use of equal weights causes large utility losses if $.6 \leq W \leq 1$ $(.28 \leq u_{y\hat{y}_e} \leq .85)$, (b) equal weights are more or less adequate if $0 \leq w \leq .6$ $(.85 \leq u_{y\hat{y}_e} \leq 1)$, (c) use of rank weights

(.67, .33 or vice versa) ensures that $u_{y\hat{y}} \geq .88$ if the rank order is correct, and (d) reversing the rank of the weights can cause very large losses (e.g., estimating a true weight of .4 by .6 yields only 64% of the maximum possible utility). For this set of alternatives, then, use of unverified equal weights is inadvisable.

Insert Figure 11 about here

The theoretical results and practical examples examined in this paper suggest that rank weights (assign importance ranks to the attributes allowing for ties, then normalize) may provide all the accuracy needed for most multiattribute decision problems. While production of an importance ordering of the attributes requires more work from the decision maker than does a priori use of equal weights, obtaining such rankings is much simpler than more traditional multiattribute procedures. A more formal evaluation of the implications of the routine use of rank weights will be the focus of a subsequent paper.

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Footnotes

1. Usually the x_{ij} are only attribute levels for each alternative and on additional function or scaling is necessary to transform the x_{ij} onto a utility scale. It is assumed throughout this paper that such scaling has already been accomplished. The x_{ij} may also be unreliably measured in practice. Because error in the estimates of weights is the primary topic of this paper, it is presumed for convenience that the true values of the x_{ij} are known.

2. A closed form analytical solution is possible for a = 2; $\rho_{x_1 x_2} = 9\pi/4 - 8 \approx -.93$, which confirms the simulation procedure. For other values of a between 1 and 10 a reasonable approximation is $\rho_{x_1 x_2} = -1.019a^{-.158}$.

Appendix: Notes on the Mathematical Proofs

Rather than giving each proof in complete detail, only the general outline of each proof is indicated. In particular algebraic reduction (which constitutes the bulk of the formal proofs) is omitted.

Theorem 1

This proof is a standard exercise in many mathematical statistics texts (e.g., Hogg & Craig, 1965, problem 4.79, p. 149).

$$\begin{aligned}\mathrm{Cov}(y\hat{y}) &= E[(y - E(y))(\hat{y} - E(y))] \\ &= E\{[w(x_1 - \mu_1) + (1 - w)\mu_2][\hat{w}(x_1 - \mu_1) + (1 - \hat{w})\mu_2]\}\end{aligned}$$

where μ_1 and μ_2 are the respective expectations for x_1 and x_2. Using basic properties of the expectation operator and the definitions of σ and ρ, this reduces to

$$\mathrm{Cov}(y\hat{y}) = w\hat{w}\sigma_1{}^2 + (1 - w)(1 - \hat{w})\sigma_2{}^2 + (w + \hat{w} - 2w\hat{w})\rho_{12}\sigma_1\sigma_2 \quad .$$

$$\mathrm{Var}(y) = \mathrm{Cov}(yy) = w^2\sigma_1{}^2 + (1 - w)^2\, \sigma_2{}^2 + 2w(1 - w)\rho_{12}\sigma_1\sigma_2 \quad .$$

The theorem assumes that x_1 and x_2 are standardized so without loss of generality, let $\sigma_1 = \sigma_2 = 1$. Substituting the above results into the definition of ρ yields the desired result.

Corollary 1.1

Let $\hat{w} = .5$ in Theorem 1.

Corollary 1.2

For $\hat{w} = 1$, $\rho_{y\hat{y}}$ is greater than $\rho_{y\hat{y}_e}$ whenever

$$1 + (w - 1)(1 - \rho) > \sqrt{.5(1 + \rho)}$$

The left side is the numerator in Theorem 1 when $\hat{w} = 1$ and the right side is the numerator in Corollary 1.1 (the denominators are equal and positive so they are cancelled from the inequality). Solving this inequality for w yields the desired result.

Footnote 2. ($\rho_{x_1x_2}$ on the unit circle in the positive quadrant)

Assume that (x_1,x_2) are uniformly distributed on the unit circle in the positive quadrant. This is equivalent to assuming that x_1^2 is uniformly distributed (with $x_2 = \sqrt{1 - x^2}$). Standard change of variable techniques give $f(x) = 2x$, $\sigma \leq x \leq 1$, as the probability density function of x_1. Therefore,

$$\mu_1 = E(x_1) = \int_0^1 2x^2\, dx = 2/3$$

$$\sigma^2 = E[x - \mu_1)^2] = \int_0^1 2(x - 2/3)^2\, x\, dx = 1/18$$

(By symmetry or direct integration, $\mu_2 = \mu_1$ and $\sigma_2^2 = \sigma_1^2$)

$$E(x_1x_2) = \int_0^1 2x^2 \sqrt{1-x^2}\, dx = [-\frac{x}{2}(1-x^2)^{3/2} + \frac{1}{4}(x\sqrt{1-x^2} + \sin^{-1} x)]_0^1 = \pi/8$$

so,

$$\rho_{x_1x_2} = \frac{E(x_1x_2) - \mu_1\mu_2}{\sigma_1\sigma_2} = \frac{9\pi}{4} - 8 \approx -.93$$

<u>Theorem 2 and Corollary 2.1</u>

These results can be obtained from Theorem 3 and Corollary 3.1, respectively, by letting k = 2 so their direct proofs are omitted here.

<u>Corollary 2.2</u>

Using $\hat{w} = 1$ implies that $\hat{y}$ is maximized when $x_1 = 1$ and $x_2 = 0$ so $y = w$. Then $u_{y\hat{y}}$ using $\hat{w} = 1$ is greater than $u_{y\hat{y}_e}$ whenever w is greater than the numerator in Corollary 2.1. That is, $w > 2^{-1/a}$.

<u>Theorem 3</u>

Because weights w_2 through w_k are equal by assumption, the maximum value of y will occur at the point $(x_1^*, x_2^*, \ldots, x_k^*)$ where $x_2^* = x_3^* = \ldots = x_k^*$. Therefore $\Sigma x_i^a = 1$ implies $y = wx + (1 - w)(\frac{1 - x^a}{k - 1})^{\frac{1}{a}}$. Setting the der' ⁻: ve of y with respect to x equal to zero yields

$$w - (1 - w)(k - 1)^{-1/a}(x^{a-1})(1 - x^a)^{\frac{1-a}{a}} = 0.$$

Solving this equation for x yields

$$x_1^* = f(w), \text{ as defined in the theorem, and}$$

$$x_2^* = \ldots = x_k^* = g(w).$$

Identical calculations using $\hat{w}$ define the point which would be selected so as to maximize $\hat{y}$. Substitution of these results into the definition of $u_{y\hat{y}}$ finishes the proof.

<u>Corollary 3.1</u>

Let $\hat{w} = \frac{1}{k}$ in Theorem 3.

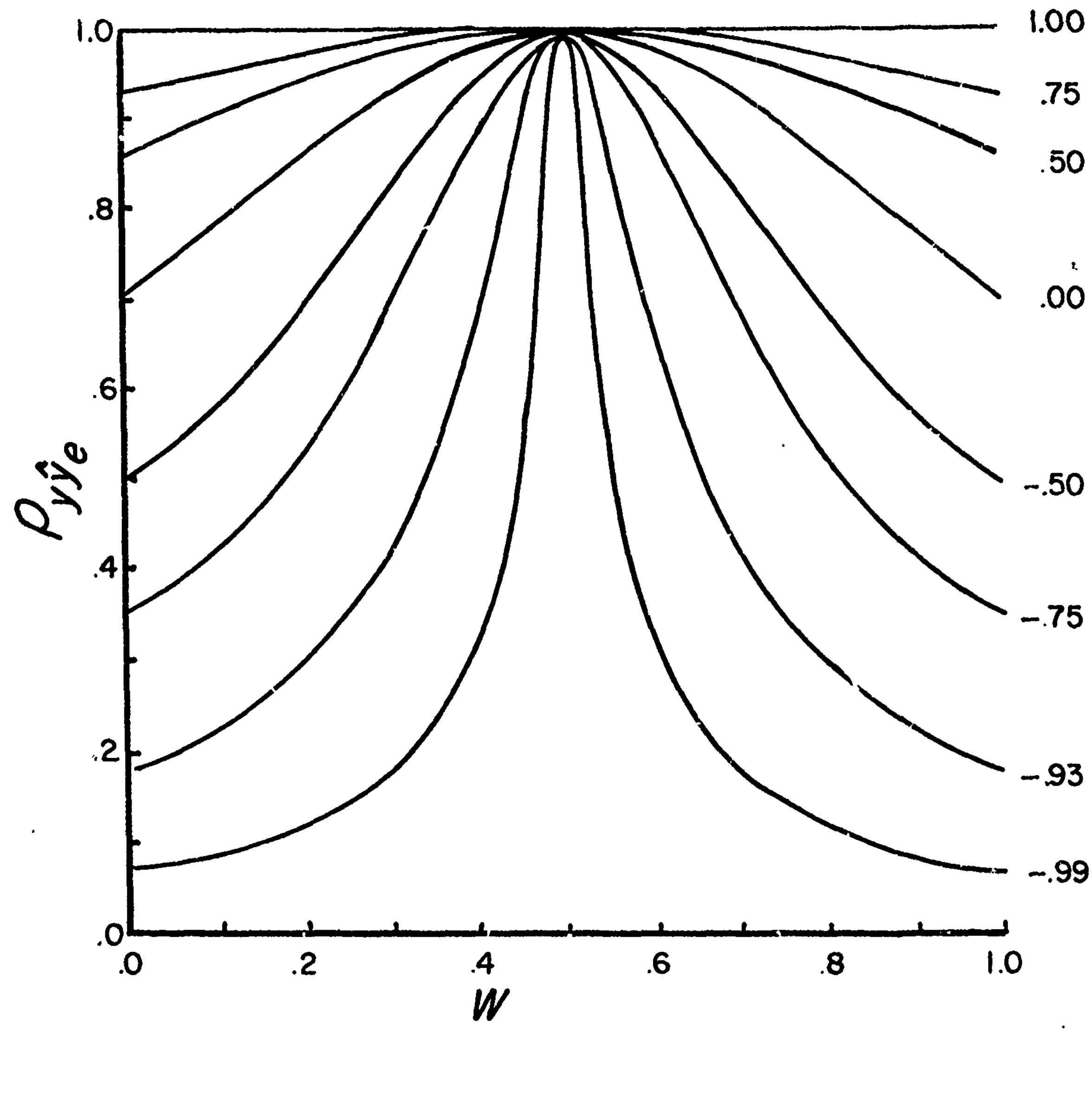
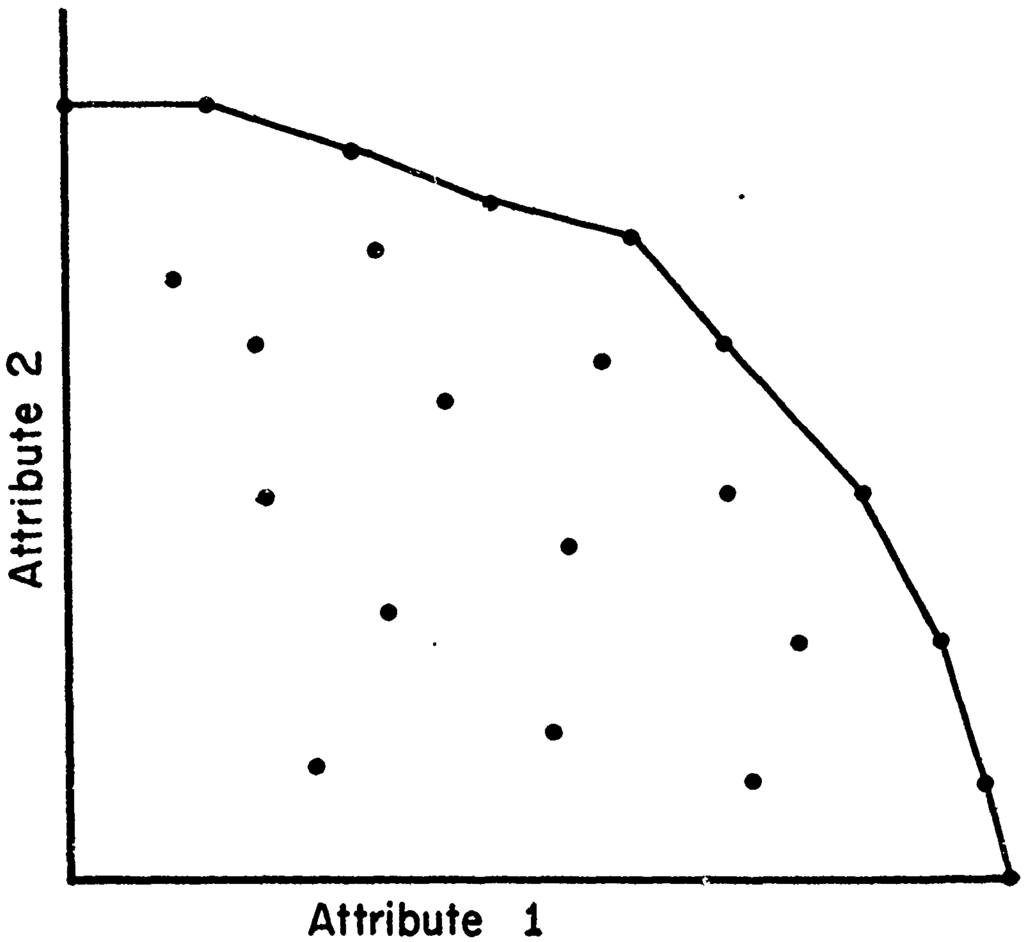


Figure 1. Correlation between true (y) and estimated ($\hat{y}$) evaluations when using equal weights as a function of the true weight w and the attribute intercorrelation. (The attribute intercorrelation $\rho_{x_1x_2}$ is at the right of each curve; all curves are symmetric about w = .5.)



F'gure 2. Illustration of Pareto boundary as non-dominated alterna ives. (Solid line is Pareto boundary; points below the line are dominated alternatives.)

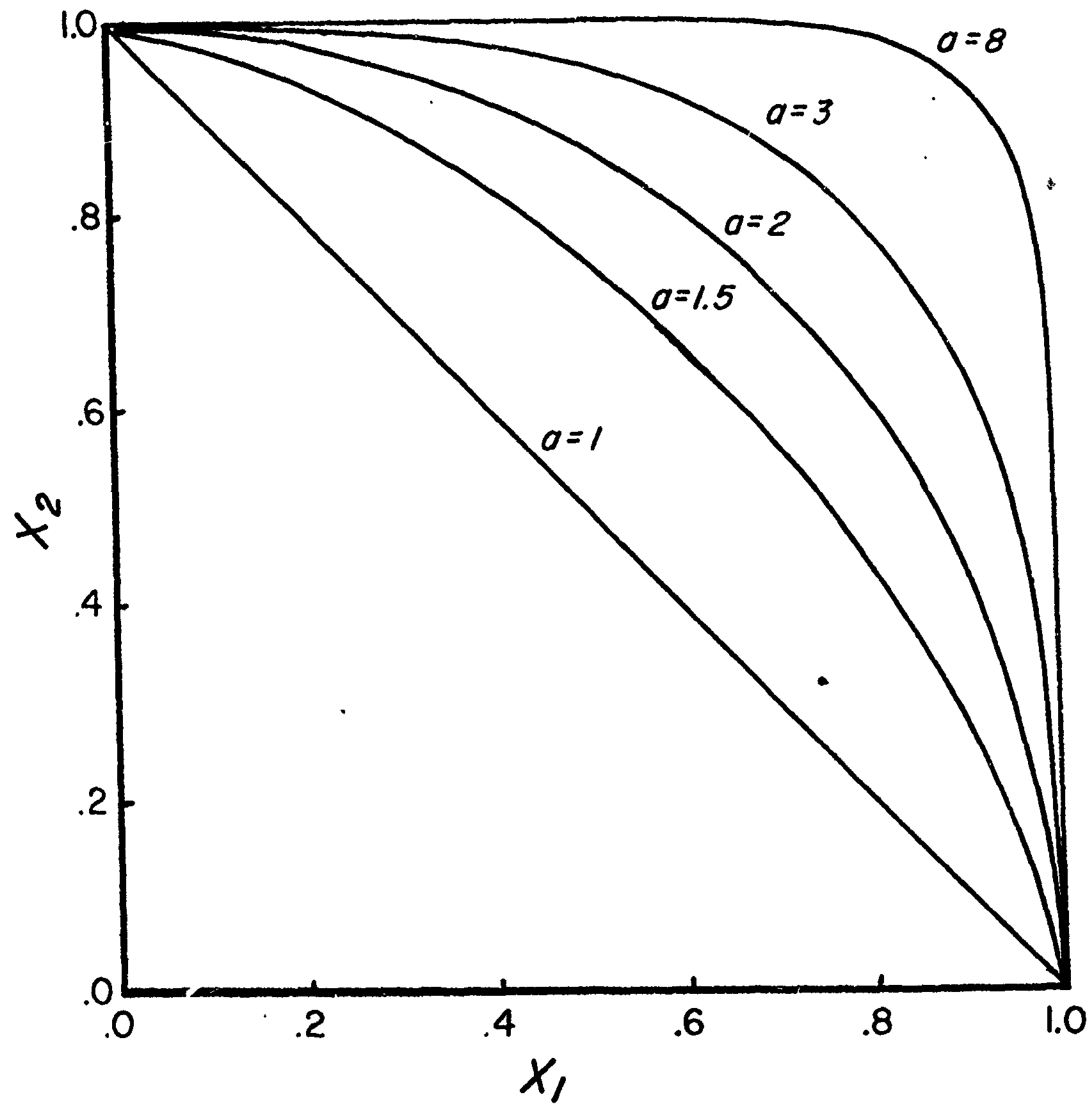


Figure 3. Prototypical Pareto curves defined by $x_1^a + x_2^a = 1$.

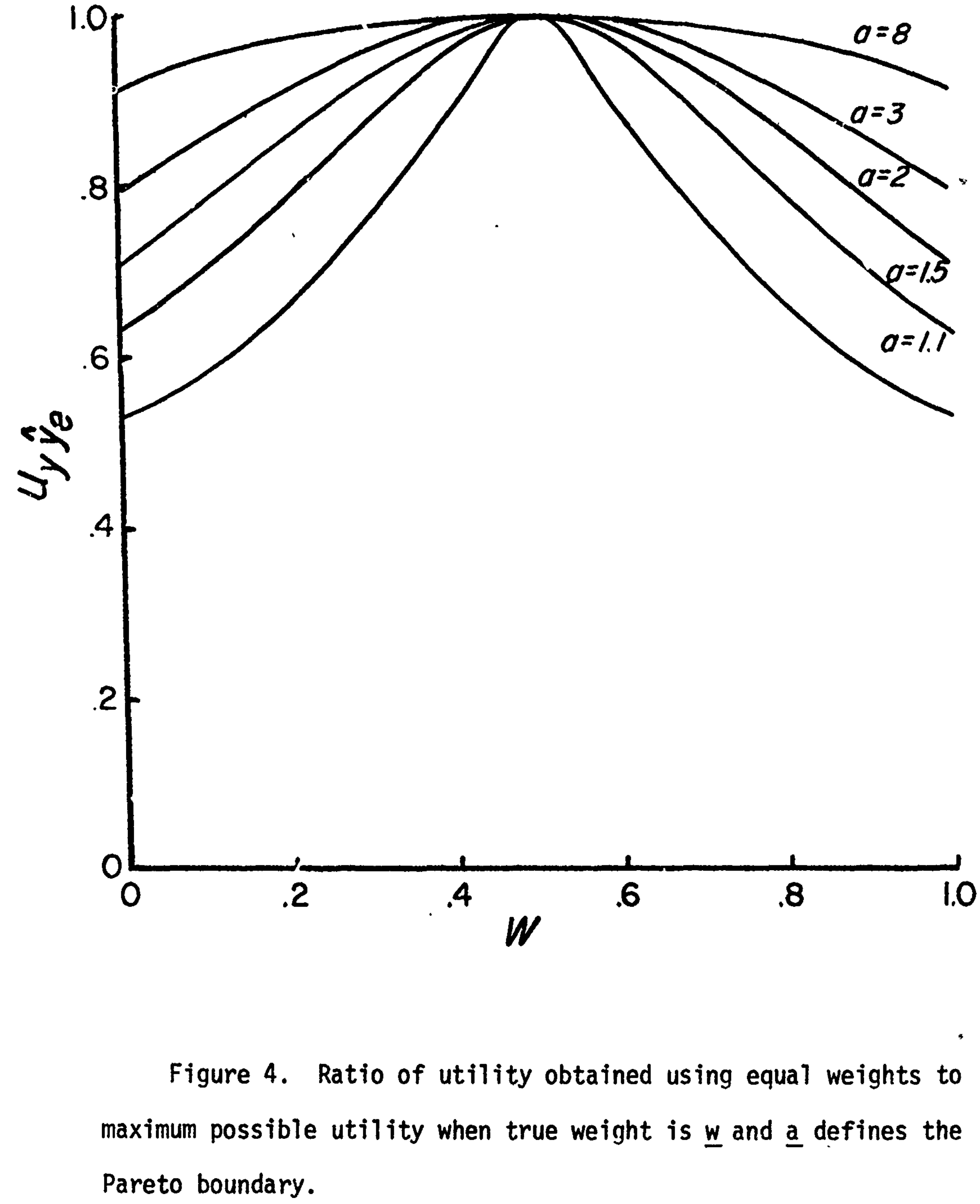


Figure 4. Ratio of utility obtained using equal weights to maximum possible utility when true weight is w and a defines the Pareto boundary.

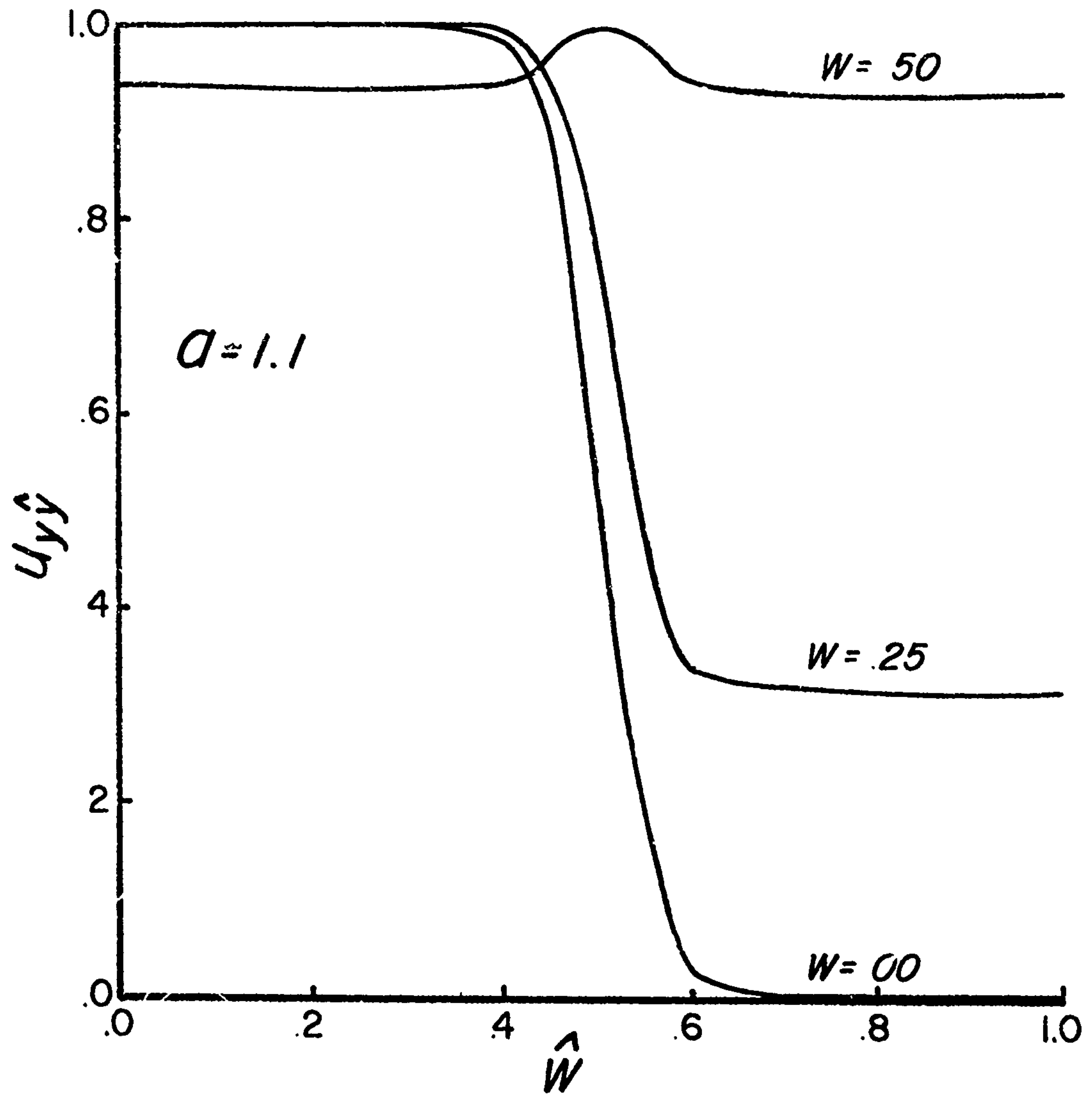


Figure 5. Utility loss caused by misestimating the true weight $\underline{w}$ with the estimate $\hat{w}$ for a severe attribute tradeoff.

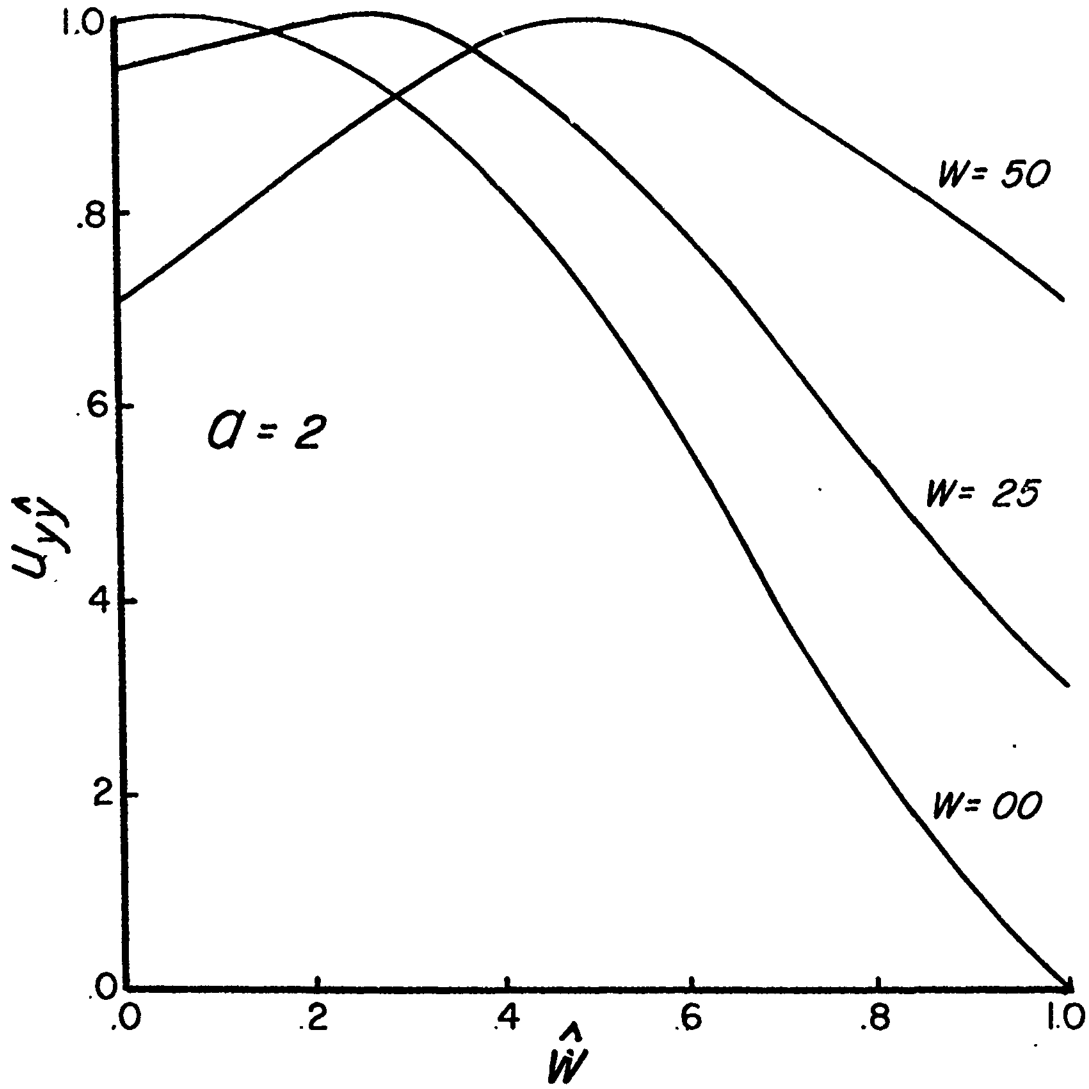


Figure 6. Utility loss caused by misestimating the true weight w with the estimate $\hat{w}$ for a moderate attribute tradeoff.

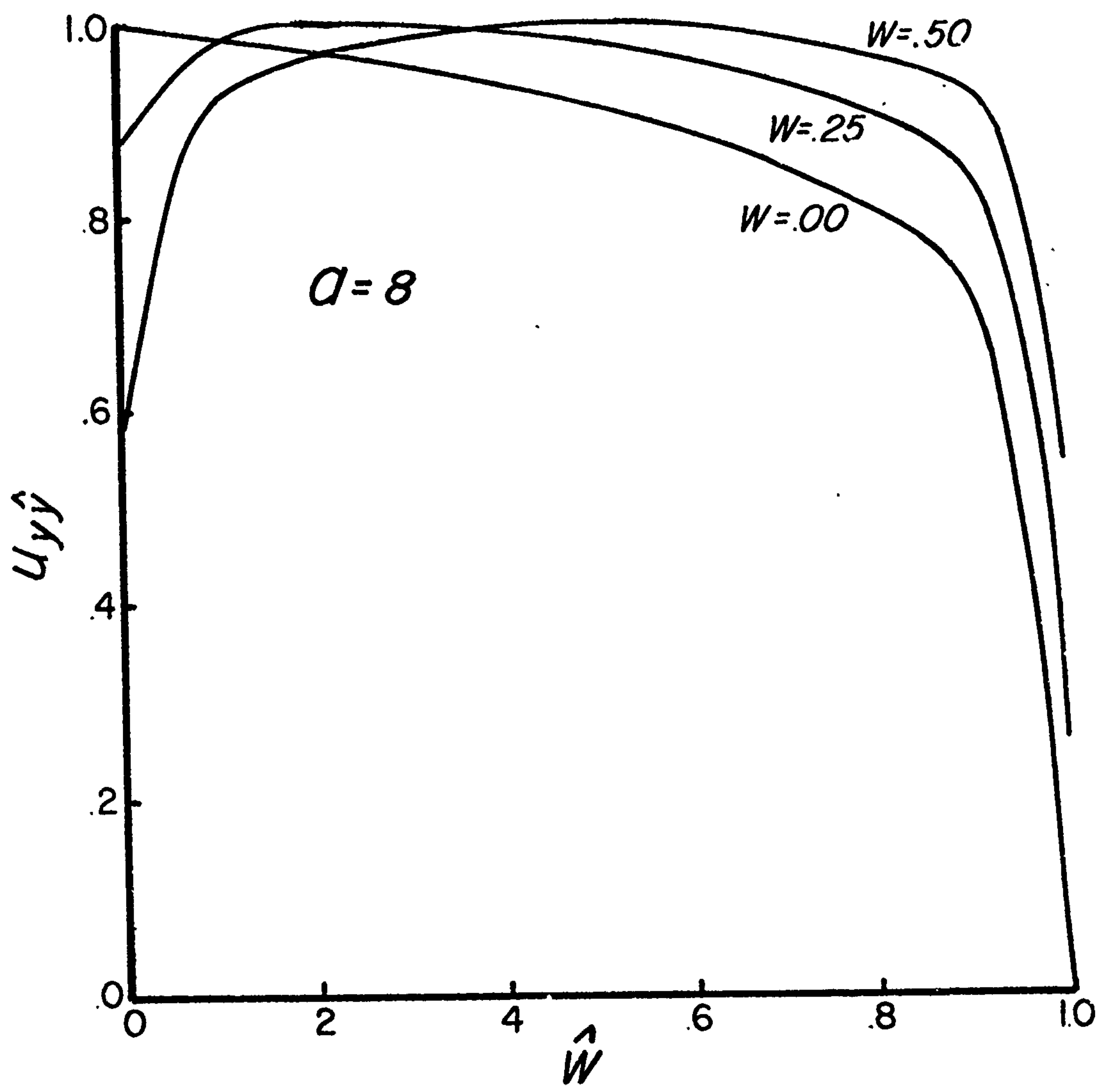


Figure 7. Utility loss caused by misestimating the true weight $\underline{w}$ with the estimate $\hat{w}$ for a weak attribute tradeoff.

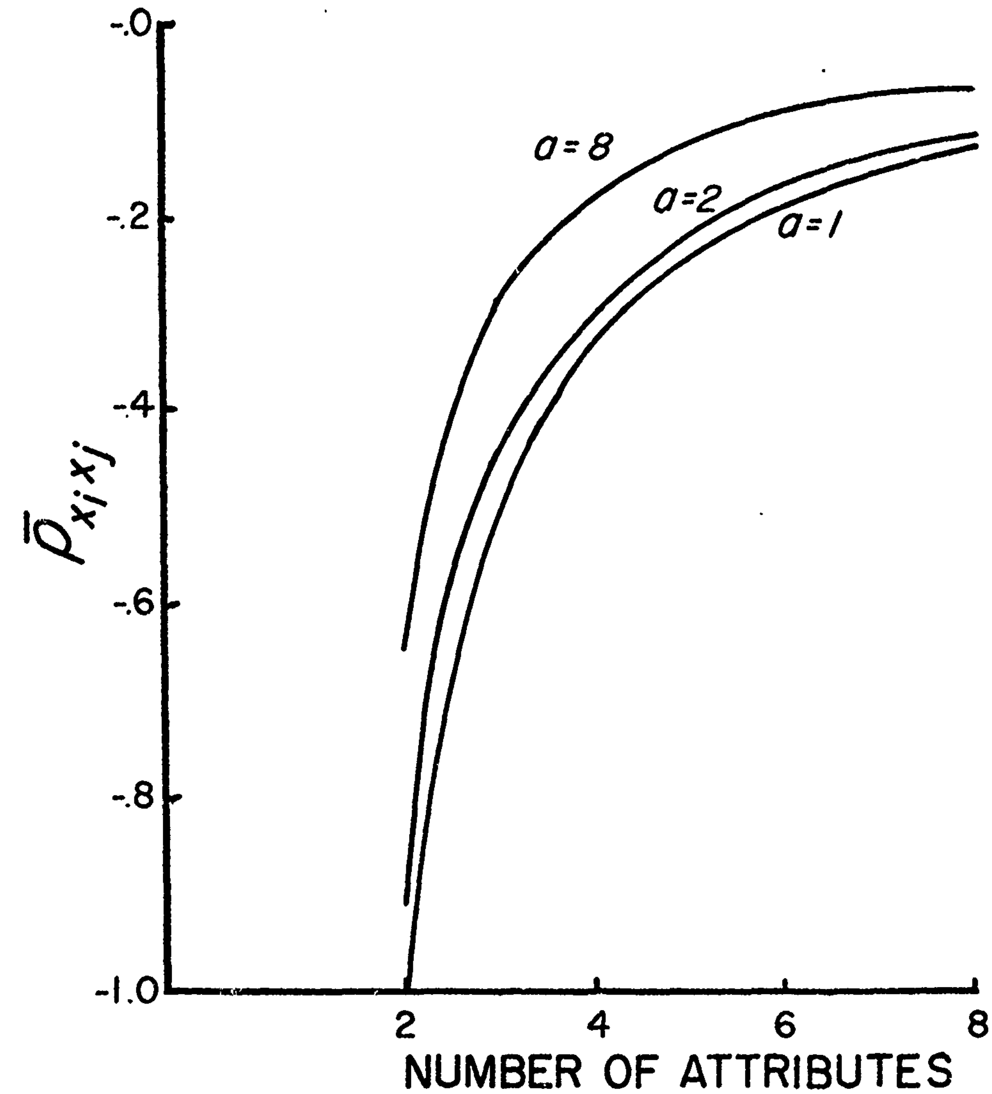


Figure 8. Average correlation between attributes as a function of the number of attributes for prototypical Pareto curves defined by $\Sigma x_i^a = 1$.

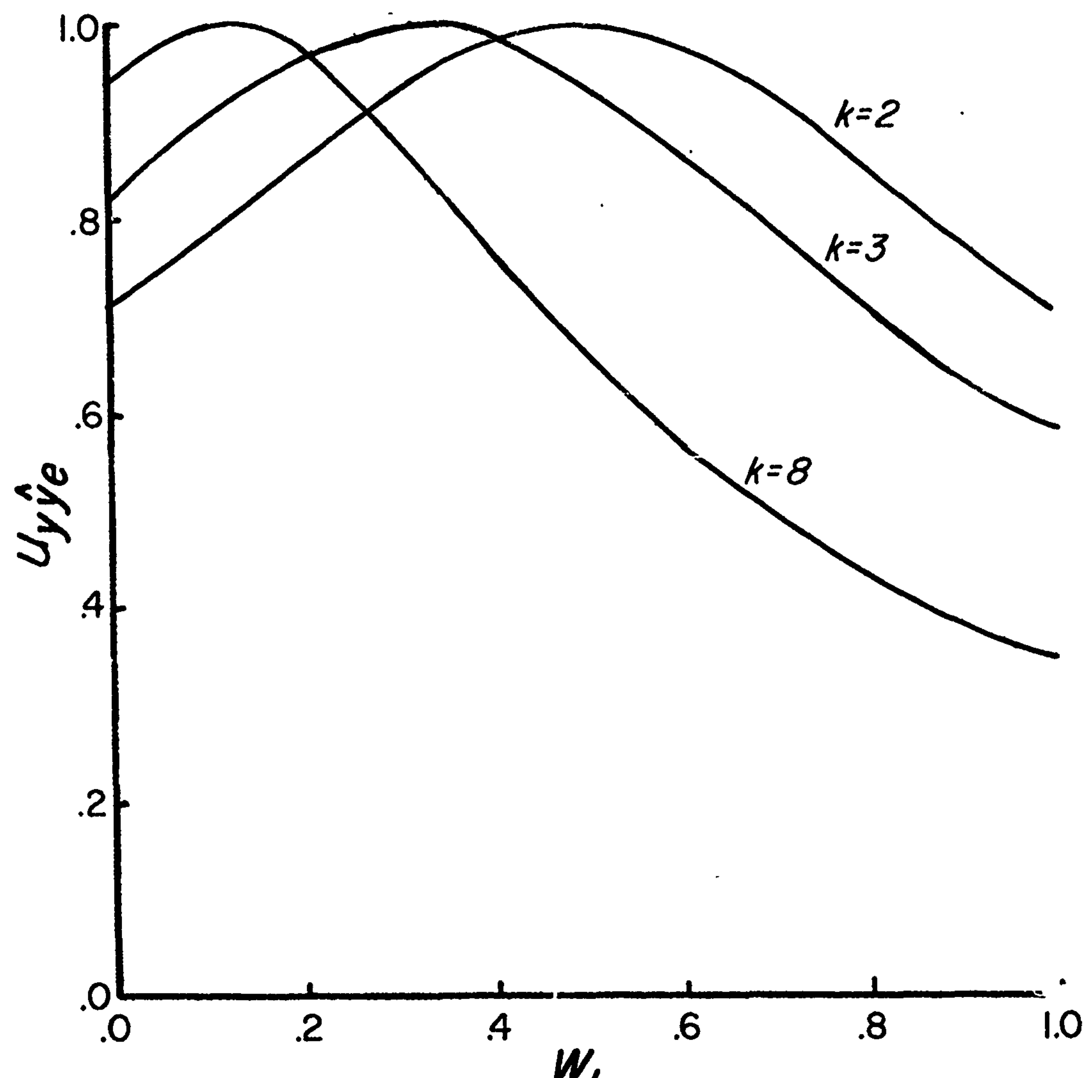


Figure 9. Ratio of utility obtained using equal weights to maximum possible utility when true weight for attribute x_1 is w_1 and all other true weights are equal for a moderate tradeoff (a = 2) for various numbers of attributes.

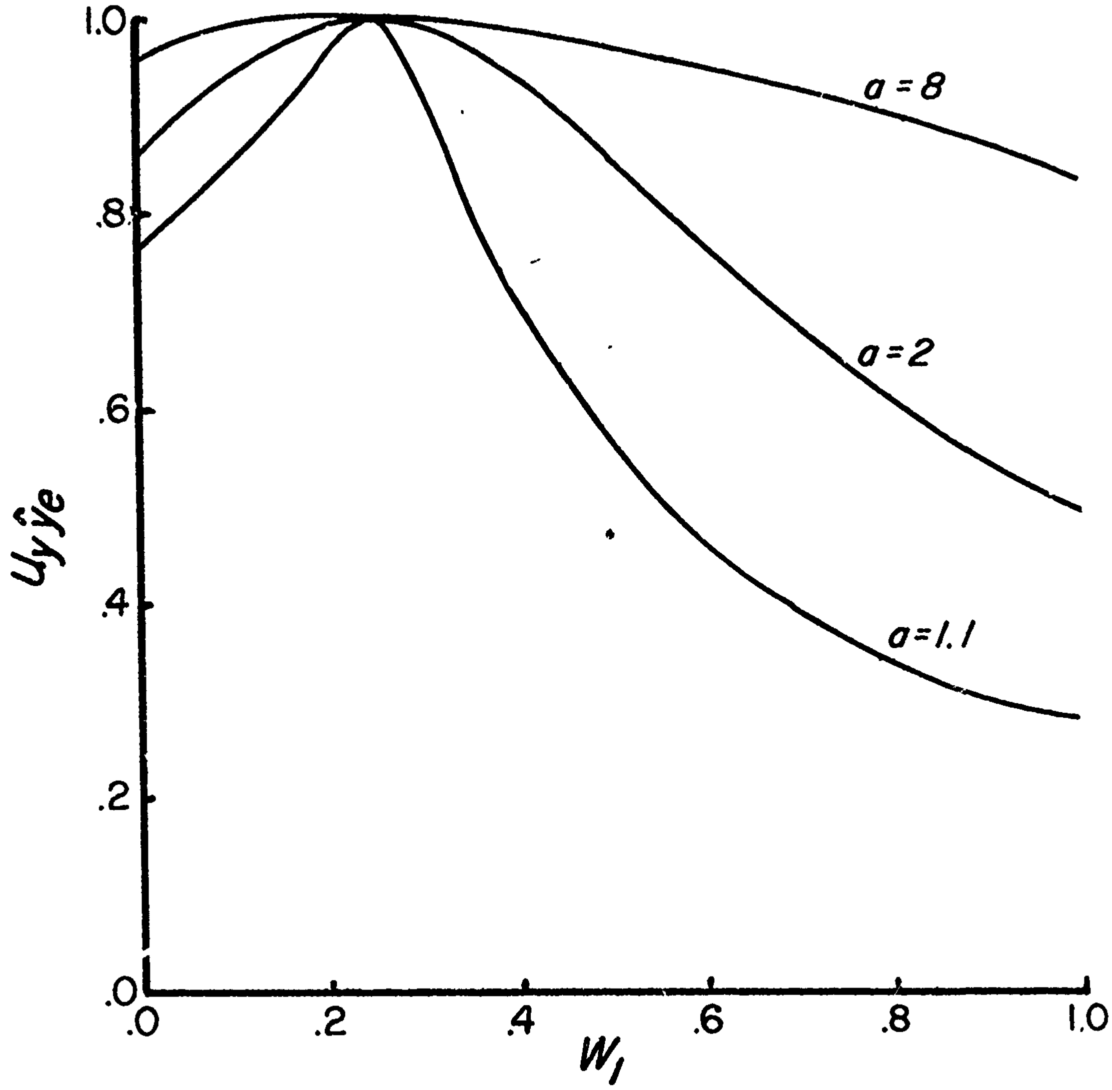


Figure 10. Ratio of utility obtained using equal weights to maximum possible utility when true weight for attribute x_1 is w_1 and all other true weights are equal for various tradeoff relationships for four attributes.

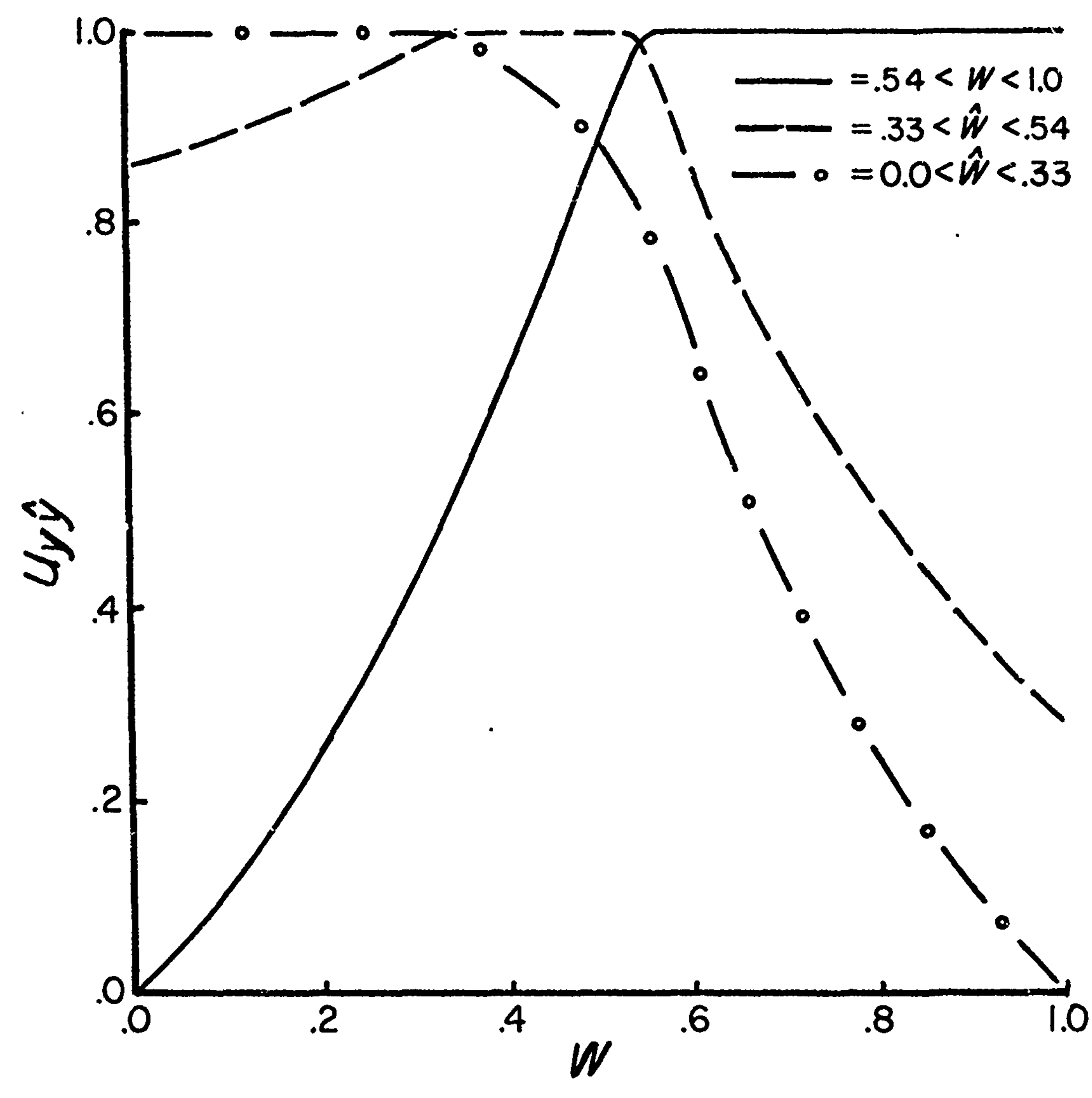


Figure 11. $u_{y\hat{y}}$ for a practical example. (Based on Hammond & Adelman, 1976.)

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